

2.33 For argon at room temperature and 1 atm, the volume per molecule is

$$\frac{V}{N} = \frac{kT}{P} = \frac{(1.38 \times 10^{-23} \text{ J/K}) \cdot 300 \text{ K}}{10^5 \text{ N/m}^2} = 4.14 \times 10^{-26} \text{ m}^3$$

the energy per volume is:

$$\frac{U}{N} = \frac{3}{2} kT = \frac{3}{2} \cdot (1.38 \times 10^{-23} \text{ J/K}) \cdot 300 \text{ K} = 6.21 \times 10^{-21} \text{ J}$$

The mass of an argon atom is  $40 \text{ u} = 6.64 \times 10^{-26} \text{ kg}$

Then

$$\left( \frac{V}{N} \right) \left( \frac{4\pi mU}{3Nh^2} \right)^{3/2} = (4.14 \times 10^{-26} \text{ m}^3) \left( \frac{4\pi (6.64 \times 10^{-26} \text{ kg}) (6.21 \times 10^{-21} \text{ J})}{3 (6.63 \times 10^{-34} \text{ J}\cdot\text{s})^2} \right) = 1.02 \times 10^7$$

Entropy:

$$S = R \left[ \ln(1.02 \times 10^7) + \frac{5}{2} \right] = 155 \text{ J/K}$$

Because Ar is heavier, which means with same energy, Ar has larger momentum than He. The larger momentum resulting a larger "hypersphere" in momentum space and corresponds to a larger multiplicity.

2.36.

1) Book:

1 Book is around 100 moles carbon, which is  $10^{26}$  particles

$$S \sim NK \sim 10^{26} \cdot 10^{-23} \text{ J/K} = 1000 \text{ J/K}$$

2) Water:

400kg water  $\sim 10^{28}$  H<sub>2</sub>O

$$S \sim NK \sim 10^{28} \cdot 10^{-23} \text{ J/K} = 10^5 \text{ J/K}$$

3)  $2 \times 10^{30}$  kg Hydrogen  $\sim 10^{56}$  particle.

$$S \sim NK \sim 10^{56} \cdot 10^{-23} \text{ J/K} \sim 10^{33} \text{ J/K}.$$

2.37. Similar to (2.53)

$$\Delta S_A = \left[ \frac{N_A}{(1-x)N} \right] k \ln \frac{V_f^{(A)}}{V_i^{(A)}} = [(1-x)N] k \ln \frac{1}{1-x} = -NK(1-x) \ln(1-x)$$

$$\Delta S_B = N_B k \ln \frac{V_f^{(B)}}{V_i^{(B)}} = [xN] k \ln \frac{1}{x} = -NKx \ln x$$

$$\Delta S_{\text{mixing}} = \Delta S_A + \Delta S_B = -NK [x \ln x + (1-x) \ln(1-x)]$$

When  $x = \frac{1}{2}$

$$\Delta S_{\text{mixing}} = -NK \left[ \frac{1}{2} \ln\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \ln\left(\frac{1}{2}\right) \right] = NK \ln 2$$

Replace  $N$  with  $2N$ , it will go back to (2.54)

2.42

a) The unit of  $G$ :  $m^3 \cdot kg^{-1} \cdot s^{-2}$

The unit of  $M$ :  $kg$

The unit of  $c$ :  $m \cdot s^{-1}$

Assume radius is a constant multiplying a combination of  $G, M, c$ .  


The unit of radius is  $m$ .

Then we assume:

$$r = \text{const} \cdot (G)^\alpha (M)^\beta (c)^\gamma$$

$$\left. \begin{array}{l} \text{For } m: \quad 1 = 3\alpha + \gamma \\ \text{kg:} \quad 0 = -\alpha + \beta \\ \text{s} \quad 0 = -2\alpha - \gamma \end{array} \right\} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = 1 \\ \gamma = -2 \end{cases}$$

So radius  $\sim GM/c^2$ .

$$\text{For } M = 2 \times 10^{30} \text{ kg} \quad \frac{GM}{c^2} = \frac{6.67 \times 10^{-11} m^3 kg^{-1} \cdot s^{-2} \cdot 2 \times 10^{30} \text{ kg}}{(3 \times 10^8 m \cdot s^{-1})^2} = 1.482 \times 10^3 \text{ m}$$

b) The entropy of a system is of the order  $N$ . Assume we start from  $N_1$  particles, and form the black hole, accounting to the 'second law', the entropy  $\geq$  order of  $N_1$ . So assume we start from  $N_1 \dots N_m \dots$ , since the final results is the same, the entropy  $\sim \sup \{N_i, i=1, 2, \dots\}$ . Thus entropy should be the order of maximum possible particles.

c). Suppose the wavelength  $\lambda = \frac{GM}{c^2}$

$$\Sigma = hc/\lambda = \frac{hc^3}{GM}$$

The total energy is given  $Mc^2$ , then the # of photons

$$N = \frac{Mc^2}{\Sigma} = \frac{GM^2}{hc}$$

$$\text{Entropy } S \sim Nk = \frac{GM^2 k}{hc}$$

$$d) S_{b.h} = \frac{8\pi^2 GM^2}{hc} \cdot k$$

$$= \frac{8\pi^2 \cdot (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}) \cdot (2 \times 10^{30} \text{ kg})^2}{(6.63 \times 10^{-34} \text{ J} \cdot \text{s}) (3 \times 10^8 \text{ m} \cdot \text{s}^{-1})} \cdot (1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1})$$

$$= 1.5 \times 10^{54} \text{ J/K}$$

As we calculated in 2.36) the entropy of sun is  $10^{33} \text{ J/K}$ .

To equal the entropy of a single one-solar-mass black holes, you need  $10^{20}$  ~~sun~~ sun.

B.3.

a)  $\int_{-\infty}^{+\infty} x e^{-ax^2} dx$ . due to the symmetry.  $\Rightarrow 0$ .

b)  $\int_{-\infty}^x \tau e^{-a\tau^2} d\tau = - \int_x^{+\infty} \tau e^{-a\tau^2} d\tau$   
 $= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-a(\tau^2)} d(\tau^2) = \frac{1}{2a} e^{-ax^2}$

c)  $\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a} \int_0^{\infty} e^{-ax^2} d(ax^2) = \frac{1}{2a}$

d)  $I(a) = \int_0^{\infty} x e^{-ax^2} dx$

$-\frac{dI(a)}{da} = \int_0^{\infty} x^3 e^{-ax^2} dx = \frac{1}{2a^2}$

B.15

a)  $\int_0^1 \int_0^1 \dots \int_0^1 e^{-(x_1^2 + x_2^2 + \dots + x_d^2)} dx_1 dx_2 \dots dx_d$   
 $= \left[ \int_0^1 e^{-x_1^2} dx_1 \right]^d = (\sqrt{\pi})^d = \pi^{d/2}$

b)  $I = \int_0^R r \int_{\phi_1=0}^{2\pi} \int_{\phi_2=0}^{\pi} \dots \int_{\phi_{d-1}=0}^{\pi} e^{-r^2} \frac{[r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2})] dr d\phi_1 \dots d\phi_{d-1}}{dr d\phi_1 \dots d\phi_{d-1}}$

where  $\int_{\phi_1=0}^{2\pi} \int_{\phi_2=0}^{\pi} \dots \int_{\phi_{d-1}=0}^{\pi} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) d\phi_1 \dots d\phi_{d-1}$   
 $= A_d(1)$ , proved by follows:

Consider a d-dimension sphere of radius R

$V_d(R) = \int_0^R \int_{\phi_1=0}^{2\pi} \int_{\phi_2=0}^{\pi} \dots \int_{\phi_{d-1}=0}^{\pi} r^{d-1} \sin^{d-2}(\phi_1) \dots \sin(\phi_{d-2}) dr d\phi_1 \dots d\phi_{d-1}$

$A_d(R) = \frac{dV_d(R)}{dR} = R^{d-1} \int_{\phi_1=0}^{2\pi} \int_{\phi_2=0}^{\pi} \dots \int_{\phi_{d-1}=0}^{\pi} \sin^{d-2}(\phi_1) \dots \sin(\phi_{d-2}) d\phi_1 \dots d\phi_{d-1}$

So plug  $R=1$  we get  $A_d(1)$ .

$$\text{Thus } I = A_d(1) \cdot \int_0^\infty r^{d+1} e^{-r^2} dr$$

$$c). \text{ Rewrite } \int_0^\infty r^{d-1} e^{-r^2} dr = \frac{1}{2} \int_0^\infty (r^2)^{\frac{d}{2}-1} e^{-r^2} d(r^2) = \frac{1}{2} \int_0^\infty t^{\frac{d}{2}-1} e^{-t} dt = \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\text{Recall from a) } I = \pi^{d/2}$$

Then we have :-

$$\pi^{d/2} = A_d(1) \cdot \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

$$A_d(1) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

$$A_d(r) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1}$$